Math 10A
Worksheet, Discussion #21; Monday, 7/16/2018
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1 Improper Integrals

1. **TRUE** False It is possible for the integral $\int_1^\infty f(x)$ to be neither a finite number nor infinity.

> **Solution:** We showed that the integral $\int_0^\infty \cos(x)dx$ doesn't exist at all in class.

2. True **FALSE** Since $3 > 1$, the integral $\displaystyle\int_0^\infty \frac{1}{x^3}dx$ converges.

> **Solution:** We only showed the result where the bottom limit is 1. This integral actually diverges.

3. True **FALSE** If $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)dx$ converges.

> **Solution:** Counter example is $f(x) = \frac{1}{x}$.

4. Calculate $\displaystyle\int_3^\infty \frac{1}{x\ln(x)}$.

> **Solution:** We have that
> $$\int_3^\infty \frac{1}{x\ln x} = \lim_{t\to\infty}\int_3^t \frac{1}{x\ln x} = \lim_{t\to\infty} \ln(\ln x)|_3^t$$
> $$= \lim_{t\to\infty}[\ln(\ln(\infty)) - \ln(\ln 3)] = \infty.$$

5. Calculate $\displaystyle\int_1^\infty e^{-5x}dx$.

Solution: We have that

$$\int_1^\infty e^{-5x}dx = \lim_{t\to\infty}\int_1^t e^{-5x}dx = \lim_{t\to\infty}\frac{e^{-5x}}{-5}|_1^t$$

$$= \lim_{t\to\infty}[\frac{e^{-5t}}{-5} + \frac{e^{-5\cdot 1}}{5} = \frac{e^{-5}}{5}.$$

6. Calculate $\int_1^\infty \frac{x}{\sqrt{x^2+1}}dx$.

Solution: We have that

$$\int_1^\infty \frac{x}{\sqrt{x^2+1}}dx = \lim_{t\to\infty}\int_1^t \frac{x}{\sqrt{x^2+1}}dx = \lim_{t\to\infty}\sqrt{x^2+1}|_1^t$$

$$= \lim_{t\to\infty}[\sqrt{t^2+1}-\sqrt{2}] = \infty.$$

7. Calculate $\int_0^\infty \frac{1}{1+x^2}dx$.

Solution: We have that

$$\int_0^\infty \frac{1}{1+x^2}dx = \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}dx = \lim_{t\to\infty}\arctan(x)_0^t = \lim_{t\to\infty}\arctan(t) = \frac{\pi}{2}.$$

2 Convergence

8. True **FALSE** If $a < b$ then $ac < bc$.

Solution: We have that $1 < 2$ but $-1 \not< -2$.

9. True **FALSE** If $a < b$, then $\frac{1}{a} > \frac{1}{b}$.

Solution: Take $-1 < 2$ but $-1 \not> \frac{1}{2}$.

10. True **FALSE** If $f \leq g$ and $\int_1^\infty g(x)dx$ converges, then $\int_1^\infty f(x)dx$ converges.

Solution: The integral $\int_1^\infty f(x)dx$ could diverge to $-\infty$.

11. True **FALSE** If we can find a function g such that $0 \leq f \leq g$, then $\int_1^\infty f(x)dx$ converges.

Solution: In order to say the integral of f converges, we need to show that the integral of g converges as well.

12. Does $\int_0^\infty \frac{\arctan^2(x)}{\sqrt{1+x^4}}dx$ converge?

Solution: First we know that $|\arctan(x)| \leq \frac{\pi}{2}$ and so $\arctan^2(x) \leq \frac{\pi^2}{4}$. Also, we know that $1 + x^4 \geq x^4$ and so $\sqrt{1+x^4} \geq \sqrt{x^4} = x^2$ and $\frac{1}{\sqrt{1+x^4}} \leq \frac{1}{x^2}$. Thus

$$0 \leq \int_1^\infty \frac{\arctan^2(x)}{\sqrt{1+x^4}}dx \leq \int_1^\infty \frac{\pi^2/4}{x^2}dx = \frac{\pi^2}{4}.$$

So this integral converges.

Then for the remaining part, we know that $1 + x^4 \geq 1$ and so $\sqrt{1+x^4} \geq 1$ and $\frac{1}{\sqrt{1+x^4}} \leq 1$ and so

$$\int_0^1 \frac{\arctan^2(x)}{\sqrt{1+x^4}}dx \leq \int_0^1 \frac{\pi^2/4}{1}dx = \frac{\pi^2}{4}.$$

Therefore

$$0 \leq \int_0^\infty \frac{\arctan^2(x)}{\sqrt{1+x^4}}dx \leq \frac{\pi^2}{4} + \frac{\pi^2}{4} = \frac{\pi^2}{2}.$$

This integral converges.

13. Does $\int_3^\infty \frac{1}{\sqrt{x}\ln(x)}$ converge?

Solution: We know that for $x \geq 3$ that $x \geq \sqrt{x}$ and so $x\ln(x) \geq \sqrt{x}\ln(x)$ so $\frac{1}{x\ln(x)} \leq \frac{1}{\sqrt{x}\ln(x)}$ so

$$\int_3^\infty \frac{1}{\sqrt{x}\ln(x)} \geq \int_3^\infty \frac{1}{x\ln(x)}dx = \infty.$$

So this integral diverges.

14. Does $\int_1^\infty e^{-5x\sqrt{x}}dx$ converge?

Solution: For $x \geq 1$, we know that $x\sqrt{x} \geq x$ so $-x\sqrt{x} \leq -x$ and so $e^{-5x\sqrt{x}} \leq e^{-5x}$ and so

$$\int_1^\infty e^{-5x\sqrt{x}}dx \leq \int_1^\infty e^{-5x}dx = \frac{e^{-5}}{5}.$$

So this integral converges.

15. Does $\int_1^\infty \frac{x}{\sqrt{x^2+1}-e^{-x}}dx$ converge?

Solution: We have that $\sqrt{x^2+1}-e^{-x} \leq \sqrt{x^2+1}$ so $\frac{x}{\sqrt{x^2+1-e^{-x}}} \geq \frac{x}{\sqrt{x^2+1}}$ and so

$$\int_1^\infty \frac{x}{\sqrt{x^2+1}-e^{-x}}dx \geq \int_1^\infty \frac{x}{\sqrt{x^2+1}}dx = \infty,$$

so this integral diverges.

16. Does $\int_0^\infty \frac{1}{(1+x^2)^2}dx$ converge?

Solution: We know that $(1+x^2) \geq 1$ and so $(1+x^2)^2 \geq (1+x^2)$ so $\frac{1}{(1+x^2)^2} \leq \frac{1}{1+x^2}$ so

$$\int_0^\infty \frac{1}{(1+x^2)^2}dx \leq \int_0^\infty \frac{1}{1+x^2}dx = \frac{\pi}{2}.$$

So this integral converges.

17. Does $\int_1^\infty \sqrt{x}e^{-2x}$ converge?

Solution: We have that $\sqrt{x} \leq x$ for $x \geq 1$ and so $\sqrt{x}e^{-2x} \leq xe^{-2x}$ and so

$$\int_1^\infty \sqrt{x}e^{-2x} \leq \int_1^\infty xe^{-2x} = \frac{e^{-2}}{2} + \frac{e^{-2}}{4},$$

so the integral converges.